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GENERAL MOTORS CORPORATION

TECHNICAL REPORT

ON

AN EXACT SOLUTION OF THE TRANSIENT TEMPERATURE DISTRIBUTION IN A SPHERICAL REGION SUBJECTED TO AN ARBITRARY HEAT FLUX

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FOREWORD

This report is one of a series of related papers covering various aspects of a broad program to investigate the flow-field variables associated with hypersonic-velocity projectiles in free flight under controlled environmental conditions. The experimental research is being conducted in the Flight Physics Range of GM Defense Research Laboratories, General Motors Corporation, and is supported by the Advanced Research Projects Agency under Contract No. DA-01-021-AMC-11359(Z). It is intended that this series of reports, when completed, will provide a background of knowledge of the phenomena involved in the basic study and thus aid in a better understanding of the data obtained in the investigation.

ABSTRACT

An exact solution to the heat conduction equation is developed for a solid spherical region subjected to an arbitrary time-independent surface heat flux. It is assumed that the thermal properties are independent of temperature and that no sources or sinks exist in the region. It is shown that the solution is valid for heated, cooled, or thermally insulated regions.

CONTENTS

NOMENCLATURE

a	radius of the sphere
a_{nm}, b_{nm}, A_{nm}, B_{nm}, C_{nmj}, D_{nmj}	expansion coefficients
C	specific heat
F	given constant
$g(\theta, \phi)$	angular dependence of heat flux
K	thermal conductivity
M	constant
P_n^m	associated Legendre function
q''_a	heat flux at surface of sphere
Q	heat generation per unit volume per unit time
r	nondimensional radius
t	nondimensional time
T	temperature
x	$\cos\theta$
y	space variable
α	thermal diffusivity
β	constant, defined by Equation (7)
ρ	density
ζ, θ, ϕ	spherical coordinates
λ_{nj}	eigenvalue
τ	time

SUBSCRIPTS

o	initial conditions
i	space coordinate
n, m, j	expansion indices

INTRODUCTION

Transient heat conduction problems in polar spheric coordinate systems are of considerable importance. One important class of boundary conditions for the spherical region is a prescribed heat flux at the surface. Since the heat flux during the initial phases of reentry is approximately independent of time, the solution to this class of problems has application to the transient heating regime of reentry vehicles. In particular, the solution of the arbitrary surface flux problem can be used to calculate the time for incipient phase change of spherical models utilized to simulate reentry in free-flight range experiments. The temperature distribution in a spherical region subjected to a constant heat flux has been reported in References 1 and 2. However, the problem of a spherical region subjected to a variable surface flux has not been investigated.

Considerable research has been conducted concerning transient temperature distributions in bodies undergoing phase changes. References 2 through 9, including the references therein, provide a relatively comprehensive bibliography of this subject. However, it is to be noted that the above authors assumed that the transient heat conduction could be represented by a one-dimensional model. When the heat flux is a function of position, various sectors of the surface will reach the phase transformation state at different times. Hence, the phase transformation is further complicated by this effect, and it is important to be able to evaluate the surface temperature distribution and time when this temperature is first reached at some point on the body.

The purpose of the present paper is to determine the exact transient temperature distribution for a spherical region subjected to a time-independent surface heat flux that may be a function of position. The solution is obtained for a material with constant thermal properties and a region without sources or sinks.

FORMULATION OF THE PROBLEM

The partial differential equation for heat conduction in an isotropic solid, including the effects of heat generation and variable thermal properties, can be written as

$$\rho C \frac{\partial T}{\partial \tau} = \nabla \cdot (K \nabla T) + Q(\zeta_i, \tau) \tag{1}$$

The coordinate system for this problem is assumed to be the region bounded by the sphere of radius a as shown in Figure 1.

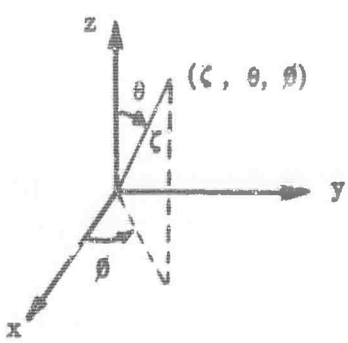

Figure 1 Coordinate System

In order to simplify the solution of Equation (1) it is assumed that: (a) the heat generation per unit volume per unit time is zero; (b) the thermal properties are constant; and (c) the boundary conditions are independent of time.

It is assumed that the initial temperature distribution is known and that an arbitrary heat flux at the surface of the sphere is specified. Utilizing the three assumptions listed above, Equation (1) can be written as

$$\frac{\partial T}{\partial \tau} = \alpha \nabla^2 T . \tag{2}$$

The solution of Equation (2) is subjected to the following initial and boundary conditions

$$T(\zeta, \theta, \phi, o) = T_o(\zeta, \theta, \phi) \tag{3a}$$

$$q''_a = \left(-K \frac{\partial T}{\partial \zeta}\right)_{\zeta = a} = F\, g(\theta, \phi) \tag{3b}$$

In Equation (3b), F is known constant and K is the thermal conductivity of the material In order to generalize the solution of Equation (2) subjected to (3a) and (3b), we define the following nondimensional distance and time

$$r = \zeta / a \tag{4a}$$

$$t = \frac{\alpha \tau}{a^2} . \tag{4b}$$

Combining Equations (2), (4a) and (4b) and expanding the Laplacian in terms of spherical coordinates yields

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} . \tag{5}$$

Converting the boundary conditions to the nondimensionalized form gives

$$T(r, \theta, \phi, o) = T_o(r, \theta, \phi) \tag{6a}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=1} = \bar{\beta}\, g(\theta, \phi) \tag{6b}$$

where

$$\bar{\beta} = -\frac{a}{K} F . \tag{7}$$

The solution of Equation (5) must also meet the physical constraint that the temperature throughout the region is bounded for finite time.

SOLUTION OF THE PROBLEM

It can be shown that a simple variables-separable solution of Equation (5) cannot meet all of the boundary conditions. This implies that if a solution with separated variables is valid for the given system, then it must be a composite function.

Assume that the solution of Equation (5) can be written in the following form

$$T(r, \theta, \phi, t) = f(t) + u(r, \theta, \phi) + v(r, \theta, \phi, t) . \tag{8}$$

It can be shown that a solution to Equation (5) that meets (6a) and (6b) is unique. A mathematical-uniqueness proof for this system is given by Churchill.[10] The physical uniqueness of the solution can also be proven by invoking the Second Law of Thermodynamics. Assume that T_1 and T_2 are unique solutions to Equation (5) subjected to (6a) and (6b). If T_1 and T_2 are solutions, then $(T_1 - T_2)$ must also be a solution. It can be easily shown that the Second Law of Thermodynamics is violated if the quantity $(T_1 - T_2)$ is not identically zero.

Combining Equations (5) and (8) yields

$$f'(t) + \frac{\partial v}{\partial t} = \nabla^2 u + \nabla^2 v . \tag{9}$$

Let

$$f'(t) = \nabla^2 u \tag{10}$$

$$\frac{\partial v}{\partial t} = \nabla^2 v . \tag{11}$$

Since the right-hand side of Equation (10) is independent of time, the left-hand side must also be independent of time. Therefore, f(t) can be written as

$$f(t) = Mt \tag{12}$$

where M is a constant. The boundary and initial conditions of the problem are satisfied by

$$T_0 (r, \theta, \phi) = u(r, \theta, \phi) + v(r, \theta, \phi, o) \tag{13}$$

$$\frac{\partial T}{\partial r}\bigg|_{r=1} = \bar{\beta}\, g(\theta, \phi) = \frac{\partial u}{\partial r}\bigg|_{r=1} + \frac{\partial v}{\partial r}\bigg|_{r=1} \tag{14}$$

Equations (13) and (14) are satisfied if we arbitrarily set

$$\frac{\partial u}{\partial r}\bigg|_{r=1} = \bar{\beta}\, g(\theta, \phi) \tag{15a}$$

$$\frac{\partial v}{\partial r}\bigg|_{r=1} = 0 \tag{15b}$$

and

$$v (r, \theta, \phi, o) = T_0 - u(r, \theta, \phi) . \tag{16}$$

Therefore, the solution to the initial problem is reduced to the solution of the following two differential equations and their corresponding initial and boundary conditions:

$$\frac{\partial v}{\partial t} = \nabla^2 v \tag{17a}$$

subjected to

$$\frac{\partial v}{\partial r}\bigg|_{r=1} = 0 \tag{17b}$$

$$v(r, \theta, \phi, o) = T_o - u(r, \theta, \phi) \tag{17c}$$

and

$$\nabla^2 u = M \tag{18a}$$

subjected to

$$\left.\frac{\partial u}{\partial r}\right|_{r=1} = \bar{\beta}\, g(\theta, \phi) \ . \tag{18b}$$

A complete solution to the initial problem is obtained when the solutions u and v are evaluated and the constant M in Equation (12) is determined.

SOLUTION OF POISSON'S EQUATION

The solution of Equation (18a) can be obtained by making the following definition:

$$u(r, \theta, \phi) = \omega(r, \theta, \phi) + \Psi(r) \tag{19}$$

where

$$\nabla^2 \omega = 0 \tag{20}$$

and

$$\Psi'' + \frac{2}{r}\Psi' = M \ . \tag{21}$$

Equation (21) is a second-order nonhomogeneous differential equation whose general solution can be written as

$$\Psi(r) = \frac{M}{6} r^2 + \frac{C_1}{r} + C_2 \ . \tag{22}$$

Since u must remain finite at the origin, the value of C_1 must be identically zero. Arbitrarily setting the value of Ψ equal to zero at the surface of the sphere, we obtain

$$\Psi(r) = \frac{M}{6}(r^2 - 1) \ . \tag{23}$$

The solution of the ω equation can be obtained by separation of variables

$$\omega(r, \theta, \phi) = R(r)\, \Theta(\theta)\, \Phi(\phi) \tag{24}$$

Combining Equations (20) and (24) and dividing by ω yields

$$\left[r^2 \frac{R''}{R} + \frac{2r R'}{R}\right](1-x^2) + \frac{(1-x^2)}{\Theta} \frac{\partial}{\partial x}\left((1-x^2)\frac{\partial \Theta}{\partial x}\right) = -\frac{\Phi''}{\Phi} = m^2 \tag{25}$$

where

$$x = \cos\theta \tag{26}$$

and

m = a separation constant.

Therefore, the solution for the Φ function can be written as

$$\Phi(\phi) = \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \qquad m = 0, 1, 2 \text{ ---- }. \tag{27}$$

Equation (25) can be rewritten as

$$r^2 R'' + 2 r R' + \beta^2 R = 0 \tag{28}$$

and

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \left[\beta^2 - \frac{m^2}{1-x^2}\right]\Theta = 0 \tag{29}$$

where

β^2 is a separation constant.

The solution to Equation (29) is given by the associated Legendre function

$$\Theta(x) = P_n^m(x) \tag{30}$$

and

$$\beta^2 = n(n+1) \qquad n = 0, 1, 2 \text{ ---} \tag{31}$$

Equation (28) is Cauchy's linear differential equation which has the solution

$$R_n(r) = C_3 r^n + C_4 r^{-(n+1)}. \tag{32}$$

For the solution to be bounded at the origin, C_4 must be identically zero. Hence the solution to Equation (19) can be written as

$$u(r, \theta, \phi) = \frac{M}{6}(r^2-1) + \sum_{n=o}^{\infty} \sum_{m=o}^{n} r^n P_n^m(x) \left\{ A_{nm} \cos m\phi + B_{nm} \sin m\phi \right\}. \tag{33}$$

Applying boundary condition (15a) to Equation (31) yields

$$\bar{\beta} g(\theta, \phi) = \frac{M}{3} + \sum_{n=1}^{\infty} \sum_{m=o}^{n} n P_n^m(x) \left\{ A_{nm} \cos m\phi + B_{nm} \sin m\phi \right\}. \tag{34}$$

Now the arbitrary function $(\bar{\beta} g(\theta, \phi))$ can be expanded in the following manner:

$$\bar{\beta} g(\theta, \phi) = \sum_{n=o}^{\infty} \sum_{m=o}^{n} P_n^m(x) \left\{ a_{nm} \cos m\phi + b_{nm} \sin m\phi \right\}. \tag{35}$$

The numerical values of the expansion coefficients a_{nm} and b_{nm} can be obtained by invoking the orthogonality relations between the trigonometric and associated Legendre functions.

The normalization integrals for the associated Legendre functions are given by

$$\int_{-1}^{1} \left[P_n^m(x) \right]^2 dx = \frac{2\,(n+m)\,!}{(2n+1)(n-m)\,!} \tag{36}$$

and the values of a_{nm} and b_{nm} are given by Equations (37) through (39).

$$a_{oo} = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \bar{\beta}\, g^* (x, \phi)\, d\phi\, d x \tag{37}$$

$$a_{nm} = \frac{(2n+1)(n-m)!}{2\pi (n+m)!} \int_{-1}^{1} P_n^m (x) \int_{0}^{2\pi} \bar{\beta}\, g(x, \phi) \cos m\phi\, d\phi\, dx \tag{38}$$

$$b_{nm} = \frac{(2n+1)(n-m)!}{2\pi (n+m)!} \int_{-1}^{1} P_n^m (x) \int_{0}^{2\pi} \bar{\beta}\, g(x, \phi) \sin m\phi\, d\phi\, dx\ . \tag{39}$$

The values of the unknown coefficients in Equation (34) are obtained by equating like terms in Equations (34) and (35). Therefore, it is seen that

$$M = 3\, a_{oo} \tag{40}$$

$$A_{nm} = \frac{a_{nm}}{n} \qquad n \neq 0 \tag{41}$$

$$B_{nm} = \frac{b_{nm}}{n} \qquad n \neq 0\ . \tag{42}$$

Since the values of a_{nm} and b_{nm} are determined in terms of the given normal boundary condition and simple quadratures, the solution of the u function is determined. It is also to be noted that the constant M (see Equations (8) and (12)) has been specified in terms of the normal boundary condition.

SOLUTION OF THE v EQUATION

The purpose of this section is to outline the solution of Equation (17a) subject to the boundary and initial conditions as given by Equations (17b) and (17c). The solution is obtained by assuming that the function is separable and by proceeding in the usual manner. This attack yields the following results:

$$v(r, \theta, \phi, t) = R(r)\, \Theta(\theta)\, \Phi(\phi)\, e^{-\lambda^2 t} \tag{43}$$

where

$$\Phi(\phi) = \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix} \qquad m = 0, 1, 2 \text{ -- } n \tag{44}$$

$$\Theta(x) = P_n^m(x) \qquad n = 0, 1, 2 \text{ --} \tag{45}$$

and

$$R'' + \frac{2}{r} R' + \left[\lambda^2 - \frac{n(n+1)}{r^2}\right] R = 0 \tag{46}$$

Equation (46) is similar to Bessel's equation, and it can be put into the correct form by the following change of variables.

Let,

$$R(r) = \frac{G(r)}{r^{1/2}} \, . \tag{47}$$

Combining Equations (46) and (47) yields

$$G'' + \frac{G'}{r} + \left[\lambda^2 - \frac{(n+1/2)^2}{r^2}\right] G = 0 \tag{48}$$

The solution to Equation (48) is given by the Bessel function of order $(n+1/2)$.

$$G(r) = J_{n+1/2}(\lambda_n r) \quad . \tag{49}$$

Combining (47) and (49) yields

$$R(r) = \frac{J_{n+1/2}(\lambda_n r)}{r^{1/2}} \quad . \tag{50}$$

Since the solution must meet the boundary condition (17b) which is independent of θ, ϕ, and t, the derivative of $R(r)$ evaluated at $r = 1$ must be zero.

$$R'(1) = \frac{d}{dr}\left[\frac{J_{n+1/2}(\lambda_{nj} r)}{r^{1/2}}\right]\Bigg|_{r=1} = 0 \quad . \tag{51}$$

It can be shown that Equation (51) is satisfied for the eigenvalues, λ_{nj}, given by

$$n J_{n+1/2}(\lambda_{nj}) = \lambda_{nj} J_{n+3/2}(\lambda_{nj}) \tag{52}$$

where

$$\lambda_{nj} > 0$$
$$n = 0, 1, 2 \text{ ---}$$
$$j = 1, 2 \text{ ---} \; .$$

The first ten eigenvalues λ_{nj} are given for n equals one to fifteen in the appendix to this report. Therefore, the solution to Equation (17a) that meets the normal boundary condition (17b) is given by

$$v(r, \theta, \phi, t) = \sum_{n=o}^{\infty} \sum_{j=1}^{\infty} \sum_{m=o}^{n} \frac{J_{n+1/2}(\lambda_{nj} r)}{r^{1/2}} P_n^m(x) \left\{ C_{nmj} \cos m\phi + D_{nmj} \sin m\phi \right\} e^{-\lambda_{nj}^2 t} \quad . \tag{53}$$

The numerical value of the coefficients C_{nmj} and D_{nmj} can be evaluated from the initial conditions as given by Equation (17c).

$$(T_o - u) = \sum_{n=o}^{\infty} \sum_{j=1}^{\infty} \sum_{m=o}^{n} \frac{J_{n+1/2}(\lambda_{nj} r)}{r^{1/2}} P_n^m(x) \left\{ C_{nmj} \cos m\phi + D_{nmj} \sin m\phi \right\} . \tag{54}$$

Since $\sin m\phi$, $\cos m\phi$, and $P_n^m(x)$ form orthogonal sets of functions and the Bessel functions are orthogonal with respect to the weight function r, the values of C_{nmj} and D_{nmj} can be evaluated in terms of quadratures. It can be shown that the value of the normalization integral for the fractional-order Bessel functions subjected to the eigenvalues of Equation (52) are given by

$$\int_0^1 r \left[J_{n+1/2}(\lambda_{nj} r) \right]^2 dr = \frac{\left[\lambda_{nj} - n(n+1) \right] \left[J_{n+1/2}(\lambda_{nj}) \right]^2}{2 \lambda_{nj}^2} . \tag{55}$$

The values of C_{nmj} and D_{nmj} are determined to be

$$\left\{ \begin{matrix} C_{nmj} \\ D_{nmj} \end{matrix} \right\} = \frac{\lambda_{nj}^2 (2n+1)(n-m)!}{\pi (n+m)! \, (\lambda_{nj}^2 - n(n+1) \left[J_{n+1/2}(\lambda_{nj}) \right]^2} \tag{56}$$

$$\int_0^1 r^{3/2} J_{n+1/2}(\lambda_{nj} r) \int_{-1}^1 P_n^m(x) \int_0^{2\pi} (T_o - u) \left\{ \begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix} \right\} d\phi \, dx \, dr .$$

The constant π must be replaced by 2π when C_{noj} is evaluated.

RESULTS

The solution is finally obtained by combining the various parts.

$$T = Mt + \frac{M}{6}(r^2-1) + \sum_{n=o}^{\infty} \sum_{m=o}^{n} r^n P_n^m(x) \left\{ A_{nm} \cos m\phi + B_{nm} \sin m\phi \right\}$$
$$+ \sum_{n=o}^{\infty} \sum_{m=o}^{n} \sum_{j=1}^{\infty} \frac{J_{n+1/2}(\lambda_{nj} r)}{r^{1/2}} P_n^m(x) \left\{ C_{nmj} \cos m\phi + D_{nmj} \sin m\phi \right\} e^{-\lambda_{nj}^2 t} \tag{57}$$

The values of the expansion coefficients M, A_{nm}, B_{nm}, C_{nmj}, D_{nmj} and the eigenvalues are given by Equations (40), (41), (42), (56), and (52).

The results of Equation (57) can be simplified if the initial temperature distribution is assumed constant.

In this case, the left-hand side of Equation (57) can be replaced by the temperature excess $(T-T_o)$, and the function (T_o-u) in the integrands of the expansion coefficients C_{nmj} and D_{nmj} can be replaced by $(-u)$.

If the heat flux and temperature distribution are axially symmetric functions, the solution and boundary conditions are not dependent on the angle ϕ. This simplification is reflected by setting the value of m identically equal to zero in Equation (57). If in addition to the above the initial temperature of the region is constant, further simplifications result in the solution.

DISCUSSION OF THE RESULTS

Some comments may be made concerning the generality of the above solution. The development in the preceeding sections has shown that the temperature distribution in a spherical region subjected to a known heat flux at the boundary can be represented in terms of known functions and quadratures. It is also important to note that the eigenvalues, λ_{nj}, of Equation (52) are not dependent on the form of the given normal heat flux distribution. Therefore, once the set of eigenvalues are determined, they are valid for all input functions $g(\theta, \phi)$.

The generality of the above solution is manifested by the expansion coefficients a_{nm} and b_{nm}. The sign and numerical value of a_{oo} indicates the character of the problem that is being studied. If the value of a_{oo} is greater than zero, there is a net flux of energy into the region and one would expect the internal energy of the region to rise with time. This is confirmed by the first term in the solution. On the other hand, the numerical value of a_{oo} is negative if there is a net flux of energy leaving the region. Physically, the negative sign implies that the internal energy of the region decreases with time. For a system in which the integral of the flux over the boundary is zero, one expects that the net internal energy of the region would remain constant. This phenomenon is also predicted by the solution, since under the above assumption the value of a_{oo} would be zero. Finally, if the heat flux is identically zero over the whole surface, all of the a_{nm}'s and b_{nm}'s are zero, and the solution reduces to the temperature history of an insulated spherical region with an initial temperature distribution.

The above discussion implies that the temperature variation given by Equation (57) is valid for a wide range of normal boundary conditions that include net heating, cooling, or an insulated boundary.

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REFERENCES

1. H.S. Carslaw, and J.C. Jaeger, Conduction of Heat in Solids, Second Edition, Oxford at Clarendon Press, New York, 1959, p. 242

2. R.F. Parisse, "Ablation of A Solid Sphere of A Low Conductivity Material," Polytechnic Institute of Brooklyn, Department of Aerospace Engineering and Applied Mechanics, PIBAL Report No. 083, Apr 1964

3. H.G. Landau, "Heat Conduction In A Melting Solid," Quart. of Applied Math., Vol. 8, 1950, pp. 81 – 94

4. T.R. Goodman, "The Heat Balance Integral and Its Application To Problems Involving A Change of Phase," Transactions of the ASME, Vol. 80, Feb 1958

5. W.D. Murray, and F. Landis, "Numerical and Machine Solutions of Transient Heat Conduction Problems Involving Melting Of Freezing. Part I. Method of Analysis and Sample Solutions," Trans. of the ASME, Journal of Heat Transfer, May 1959, p. 106

6. D. Baer and A. Ambrosio, "Heat Conduction In A Semi-Infinite Slab With Sublimation at the Surface," Ballistic Missiles and Space Technology, Vol. 2, Pergamon Press, New York, 1961

7. C. Economos, "Results of Ablation Tests on Several Plastic Models In A Hypersonic Wind Tunnel," Polytechnic Institute of Brooklyn, Dept. of Aerospace Engineering and Applied Mechanics, PIBAL Report No. 606, Mar 1961

8. J.E. Sunderland and R.J. Grosh, "Transient Temperature In A Melting Solid," Transactions of the ASME, Journal of Heat Transfer, Nov. 1961, p. 409

9. B.A. Boley, "A Method of Heat Conduction Analysis of Melting and Solidification Problems," Journal Math. and Physics, Vol. 40, Dec 1961, pp. 300 – 313

10. R.V. Churchill, Fourier Series and Boundary Value Problems, Second Edition, McGraw Hill Inc., New York, 1963, p. 250

APPENDIX

FIRST TEN EIGENVALUES FOR n EQUALS ONE TO FIFTEEN

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 10     FIRST 10 EIGENVALUES FOR   = 0,1,2,.....15
 N=0            +44934325+01+77252684+01+10904199+02+14066113+02+17220751+02
 +20371386+02+23519482+02+26666210+02+29811572+02+32956347+02
 N=1            +20815673+01+5940374[illegible]01+92058377+01+12404468+02+15579242+02
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 N=12           +14258325+02+19270297+02+23195006+02+26837515+02+30339[illegible]26+02
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 N=13           +15310864+02+20406585+02+24382040+02+28062145+02+31593688+02
 +3303587[illegible]+02+38417952+02+41757424+02+45065265+02+48349022+02
 N=14           +16360680+02+21537115+02+25561[illegible]74+02+29278732+02+32839777+02
 +363[illegible]5993+02+39708552+02+43065683+02+46388927+02+49686105+02
 N=15           +17408044+02+2266249[illegible]+02+26735168+02+30488006+02+34077861+02
 +37568007+02+40990923+02+44365655+02+477[illegible]29+02+51015184+02
```

$\lambda_{n,q}$ given for $n = 0, 1, 2, \ldots 15$ $\quad q = 1, 2, 3, \ldots 10$

where:

λ_{nq} are the set of eigenvalues > 0 to the following equation.

$$n\, J_{n+1/2}(\lambda_{nq}) = \lambda_{nq}\, J_{n+3/2}(\lambda_{nq})$$

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(Unclassified)

1. Heat Transfer – Mathematical analysis
2. Re-entry aerodynamics – Mathematical analysis
3. Re-entry aerodynamics – Simulation
4. Spheres – Temperature factors

I. DA-01-021-AMC-11359(Z)
II. TR65-01M
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